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# *UNIQUE COMMON FIXED POINT THEOREM IN CONE METRIC SPACES*

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**ABSTRACT**

The established fixed point theorems for self maps of complete metric spaces by altering the distances between the points with the use of a positive real valued function. In this paper, we prove a unique common fixed point theorem in cone metric spaces without appealing to continuity and commutativity conditions. Our results generalize several well-known comparable results in this literature.

**KEYWORDS**: Common fixed Point; Altering function ; Cone metric space; Coincidence points.

## INTRODUCTION

Huang and Zhang introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtained some fixed point results with the assumption that the cone is normal. Subsequently, Abbas and Jungck and Abbas and Rhoades have studied common fixed point theorems in cone metric spaces. Recently, Stojan Radenović has obtained coincidence point result for two mappings in cone metric spaces which satisfies new contractive conditions. In this paper,we prove coincidence point results in cone metric spaces which satisfy generalized contractive condition without appealing the continuity and commutativity conditions [1-3] and [5]. In all that follows B is a real Banach Space, and θ' denotes the zero element of B. For the mapping f, g: X→X, let C (f, g) denote the set of coincidence points of f and g, that is C(f, g) = {z ∈X : fz = gz}.

## PRELIMINARIES ON COMMON FIXED POINT THEOREM AND CONE METRIC SPACES

**Definition 1.1.** Let B be a real Banach Space and P a subset of B .The set P is called a cone if and only if:
(a).P is closed , non –empty and P≠ { θ }
(b). a,b∈R, a,b>=0, x,y ∈ P implies ax+by ∈P;
(c). x∈P and –x ∈P implies x = θ

**Definition 1.2.** Let P be a cone in a Banach Space B, define partial ordering '≤' with respect to P by x y if and only if y-x ∈ P.We shall write x<y to indicate x ≤ y
But x ≠ y while x << y will stand for y-x ∈ P, where Int P denotes the interior of the set P. This cone P is called an order cone.

**Definition 1.3.** Let B be a Banach Space and P ⊂B be an order cone .The order cone P is called normal if there exists K>0 such that for all x,y ∈ B,
$x \le y$ implies $\|x\| \le K\|y\|$.
The least positive number K satisfying the above inequality is called the normal constant of P.

**Definition 1.4.** Let X be a nonempty set of B .Suppose that the map
d: X x X →B satisfies :
(d1). θ ≤ d (x,y) for all x, y ∈ X and
d(x, y) =θ if and only if x = y ;
(d2).d(x, y) = d(y, x) for all x, y∈ $X$ ;
(d3).d(x, y) ≤ d(x, z) +d(y, z) for all x, y, z∈ $X$ .
Then d is called a cone metric on X and (X, d) is called a cone metric space.
The concept of a cone metric space is more general than that of a metric space.

**Definition 1.5.** Let (X, d) be a cone metric space .We say that {xn} is
(i) a Cauchy sequence if for every c in B with c $\gg\theta$ ,there is N such that
for all n, m>N, d(xn, xm) $\ll$c ;
(ii) a convergent sequence if for any c $\gg\theta$ ,there is an N such that for all
n>N, d(xn, x) $\ll$c, for some fixed x in X .We denote this xn $\rightarrow$ x (as n $\rightarrow\infty$) .

**Lemma 1.6.**Let (X,d) be a cone metric space , and let P be a normal cone with normal constant K .Let {xn } be a sequence in X .Then
(i). {xn } converges to x if and only if d(xn,x) $\rightarrow$0 (n $\rightarrow\infty$).
(ii). {xn } is a Cauchy sequence if and only if d (xn, xm)$\rightarrow$0 (n,m$\rightarrow\infty$).

## MAIN RESULTS

Now, we prove existence of coincidence point and a common fixed point theorem in cone metric spaces without appealing to continuity and commutative conditions, which generalizes the results.

The following theorem generalizes the Theorem 2.1

**Theorem 2.1.** Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose that the mappings f, g: X$\rightarrow$X are such that for some constant $\lambda \in$ (0,1) and for every x, y$\in$X are two self-maps of X satisfying

$$\|d(fx, fy)\| \leq \lambda \|d(gx, gy)\| \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

If the range of g contains the range of f and g(X) is a complete subspace of X, then f and g have coincidence point. Then, f and g have a unique common fixed point in X.

**Proof:** Let x0 be an arbitrary point in X, and let x1$\in$X be chosen such that y0 = f (x0 ) = g(x1 ). Since f(X) $\subseteq$ g(X). Let x2 $\in$ X be chosen such that y1 = f (x1 ) = g(x2 ). Continuing this process, having chosen xn$\in$ X, we chose $x_{n+1} \in$ X such that yn = f (xn ) = g($x_{n+1}$).

We first show that

$$\|d(y_n, y_{n-1})\| \leq \lambda \|d(y_{n-1}, y_{n-2})\| \text{ for } n = 2,3,\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Indeed,

$$\|d(y_n, y_{n-1})\| = \|d(fx_n, fx_{n-1})\| \leq \lambda \|d(gx_n, gx_{n-1})\| .$$

(2) implies that

$$\|d(y_n, y_{n-1})\| \leq \lambda \|d(y_{n-1}, y_{n-2})\| \leq \ldots\ldots \leq \lambda_{n-1} \|d(y1,y0)\| \ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Now we shall show that {yn} is a Cauchy sequence. By the triangle inequality,
for n>m we have

$$d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \ldots\ldots + d(y_{m+1}, y_m) .$$

Hence, as p is a normal cone,

$$\|d(y_n, y_m)\| \leq K \|d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \ldots\ldots + d(y_{m+1}, y_m)\| .$$
$$\leq K(\|d(y_n, y_{n-1})\| + \|d(y_{n-1}, y_{n-2})\| + \ldots\ldots + \|d(y_{m+1}, y_m)\|).$$

Now by (3),

$$\|d(y_n, y_m)\| \leq K(\lambda_{n-1} + \lambda_{n-2} + \ldots\ldots\ldots + \lambda_m) \|d(y1, y0)\| .$$

$$\leq \frac{K\lambda m}{1-\lambda} \|d(y1, y0)\| \rightarrow 0 \text{ as } m \rightarrow \infty.$$

From ([3], Lemma 4) it follows that {yn} is a Cauchy sequence. Since g(X) is complete, there exists a q in g(X) such that yn $\rightarrow$q as n$\rightarrow\infty$. Consequently, we can find p in X such that g(p) = q. We shall show that f(p) = q. From (1)

$\|d(gxn , fp)\| = \|d(fxn-1 , fp)\| \leq \lambda \|d(gxn-1, gp)\|$ ,
$\Rightarrow \|d(gp, fp)\| \leq \lambda \|d(gp, gp)\| = 0$.
That is, $\|d(gp, fp)\| = 0$.
Hence,
gp =q= fp , p is a coincidence point of f and g……………..(4)
Now using (1),
$d(p,gp) \leq d(p, yn) + d(yn, gp)$(by the triangle inequality)
$= d(p, yn) + d(fxn, fp)$
$\leq d(p, yn ) + \lambda\, d(gxn, gp)$
$\leq d(p, fxn)+\lambda d(gxn, gp)$ as $n\to\infty$
$\leq d(p, p) + \lambda\, d(p, gp)$
$\leq \lambda\, d(p,gp)$
$< \lambda\, d(p,gp)$ ( since, $\lambda<1$))
$= d(p, gp) < d(p, gp)$, which is a contradiction .
Therefore, d(p, gp)=0.
$\Rightarrow p = gp$.
Now,
$d(fp,p) = d(fp, gp)$
$= d(fp, fp)$ (since fp=gp)
$\leq \lambda\, d(gp, gp)$
$= \lambda\, d(p, p) = 0$ (by (1))
$\leq d(fp, p) = 0$
$\leq fp = p$
Since, fp = gp.
Therefore, fp = gp = p, f and g have a c ommon fixed point.
Uniqueness, let p1 be another common fixed point of f and g, then
$d(p, p1) = d(fp, g\, p1)$
$= d(fp, f\, p1)$
$\leq d(gp, g\, p1)$ (by (1))
$\leq d(p, p1) \leq 0$.
Therefore, d(p, p1) = 0.
$\Rightarrow p = p1$.
Therefore, f and g have a unique common fixed point.

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